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RESEARCH ARTICLE

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Optimal control in infinite horizon problems: a Sobolev space approach

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Abstract In this paper, we make use of the Sobolev space $W^{1,1}$ $(\mathbb{R}_+, \mathbb{R}^n)$ to derive at once the Pontryagin conditions for the standard optimal growth model in continuous time, including a necessary and sufficient transversality condition. An application to the Ramsey model is given. We use an order ideal argument to solve the problem inherent to the fact that L^1 spaces have natural positive cones with no interior points.

Keywords Optimal control · Sobolev spaces · Transversality conditions · Order ideal

JEL Classification Number C 61

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1 Introduction

Typically, the first-order necessary conditions of optimization problems in continuous time, the so-called Pontryagin conditions, are established using variational methods. This kind of methods is for example used throughout the textbooks of Hadley and Kemp (1973) and Kamien and Schwartz (1991), but it is indeed at the basis of optimal control theory as initially designed by Pontryagin et al. (1962). For a finite time horizon, the set of Pontryagin conditions include optimality conditions with respect to the control, state and co-state variables, plus the corresponding transversality conditions which depend on the assumptions on the time horizon and the terminal state. All these conditions can be identified using standard variational methods.

When the optimization time horizon goes to infinity, things become much more complicated. In particular, it turns out that while the usual Pontryagin conditions obtained for finite horizons with respect to the control, state and co-state variables are preserved, the transversality conditions cannot be safely extrapolated. As the horizon gets to infinity, it is quite easy to show (see for example Halkin 1974) that taking the limits of the transversality conditions obtained for finite time horizons is highly misleading. In particular, the traditional "economic" condition according to which the shadow price should go to zero as the time horizon goes to infinity was shown to be clearly erroneous in the case of non-discounted problems.

This has lead to a kind of split in the optimal control treatment under infinite horizons: while the Pontryagin conditions can still be obtained by variational methods, the transversality condition is obtained using another type of argument. This is for example true in the seminal paper of Michel (1982), who concentrates on the necessary transversality condition part. Michel provides *a fairly* general inspection into this issue in the case of discounted problems (without *a priori* sign or concavity assumptions on the objective and state functions). In such a framework, he proves that the right necessary transversality condition when time tends to infinity is the limit of the maximum of the Hamiltonian going to zero. This extends the property valid in a finite horizon problem with free terminal time to the infinite horizon case. On the other hand, he shows that this necessary condition implies the traditional "economic" transversality condition, mentioned above, provided the objective function is non-negative and if "enough possibilities of changing the state's speed exist indefinitely". Ye (1993) extends this analysis by allowing for the non-differentiability of the problem data and obtains the maximum principle in terms of differential inclusions in analogy to the finite horizon problem.

Unfortunately the resulting characterization of the cases where the "economic" transversality condition holds reveals unpractical. Alternative duality-based theories for discounted problems were developed starting with Benveniste and Scheinkman (1982). Under some concavity conditions (needed to apply an envelop condition), Benveniste and Scheinkman (1982) establishes the necessity of the transversality condition, $\lim_{t\to\infty}\left[-v_2\left(x(t),\dot{x}(t),t\right)\right]x = 0$ for the continuous time reduced form model:

$$\max \qquad \int_0^\infty v\left(x(t),\dot{x}(t),t\right)\mathrm{d}t$$

subject to

$$x(0) = x_0, \left(x(t), \dot{x}(t)\right) \subset (\mathbb{R}^n)^2,$$

when the assumptions of non-negativity and integrability of v for all feasible paths are verified. Kamihigashi (2001) generalizes this analysis by allowing for unbounded v with the assumptions of local boundedness of v_1 and v_2 and the existence of an open set that the optimal pair $\left(x^*(t), \dot{x}^*(t)\right)$ belongs to and under which $v(.,.,t)$ is continuously differentiable. Long and Shimomura (2003) prove the necessity of a transversality condition of the form

$$\lim_{t\to\infty} \left[x^*(t) - x_0\right] \left[v_2 \left(x(t), \dot{x}(t), t\right)\right] = 0,$$

under the assumption that v is twice differentiable and the optimal pair belongs to the interior of a set under $(\mathbb{R}^n)^2$.

This paper provides a simple and unified functional analysis argument to derive **at once** the Pontryagin conditions, including the transversality condition in infinite horizon problems. More specifically, we make use of the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$, which appears to be quite natural to derive not only the convenient transversality conditions, but also the whole set of Pontryagin conditions. Our choice of the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is relevant for many optimal growth models, e.g. the Ramsey model, in which the feasible capital paths are proved to belong to this space and the feasible consumption paths belong to L^1 (see Askenazy and Le Van 1999, p. 42). In addition to this crucial topological choice, our setting is based on an assumption (Assumption 4 in the text), which is close to the concept of supported control trajectories traditionally used in the optimal control literature (see for example, Peterson 1971). Combining this concept with the Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ turns out to be a powerful tool to get through the problem very easily. In particular, we extract the usual transversality conditions as necessary optimality conditions, together and at the same time as the other Pontryagin conditions.

To our knowledge, the first analysis that uses Sobolev spaces in economics was Chichilnisky (1977). She studies the problem of existence and the characterization of the solutions of optimal growth models in many sector economies. In this context, the prices are continuous linear functionals defined on the space of consumption paths. Mathematically, the question turns out to be the existence of an appropriate continuous linear functional separating the set of feasible paths from the set of paths which yield higher utilities than the optimal one. In Chichilnisky (1977), the space of consumption paths on which the optimization is performed is the completion of L^∞endowed with L^2 norms while the space of admissible capital paths is the completion of the space C_b^1 of continuously differentiable and bounded functions endowed with the norm given by the scalar product:

$$(f, g) = \int_0^\infty \left(\sum_0^k D^k f(t) D^k g(t)\right) \mathrm{e}^{-rt} \mathrm{d}t.$$

The basic tool needed to prove the existence of competitive prices for optimal programs, the Hahn-Banach theorem, requires one of the convex sets being separated to have an interior or an internal point. However, all L^p spaces with $1 \leq p < \infty$ have natural positive cones with no interior or internal points. To overcome this problem, the objective function being maximized is shown to be continuous in weaker L^2 topology. Another inconvenient feature of L^2 spaces is related to the fact that their topology is weaker. It creates a difficulty in having conditions on the utility function which yield $L^2 - continuity$ of nonlinear objective functional, the discounted social utility of the stream of consumption.

As mentioned above, in contrast to the previous studies, we shall use Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$. Nonetheless, as in the alternative approaches listed above, we still face the problem that the involved L^1 spaces have natural positive cones with no interior or internal points. In order to overcome this problem, we shall use the concepts of properness and order ideal. The notion of properness is proved to be very useful in analyzing the existence of equilibrium in Banach lattices or Riesz spaces (see the excellent survey of Aliprantis et al. 2002, and its references). The properness is a notion weaker than continuity. A complete characterization for strictly increasing separable concave functions in L^p_+ is given in Araujo and Monteiro (1989). Le Van (1996) characterizes properness for separable concave functions in L^p_+ without assuming monotonicity. Dana et al. (1997) provides an existence theorem when the consumptions sets being the positive orthant of a locally convex solid Riesz space has an empty interior. They use the approach of Mas-Colell and Zame (1991) by considering an economy restricted to the order ideal generated by the total resource, which is dense in the initial consumption space. This suffices to obtain a quasi-equilibrium price which can be extended to a linear form in the initial topology by the properness of the every utility function.

The paper is organized as follows. Section 2 presents the considered optimization problem, and gives some preliminary definitions and assumptions needed to derive our necessary and sufficient transversality condition. Section 3 proves the latter condition in the described mathematical framework, yielding the main result in Theorem 1. Section 4 is an application to the Ramsey model.

2 Preliminaries

Let $C^1_c(\mathbb{R}_+, \mathbb{R}^n)$ denote the set of continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}^n$ with compact support. We have the following general definitions and notations.

Definition 1 *The space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of functions, $x \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ such that there exists a function $x' \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ that satisfies*

$$\int_0^\infty x\phi' \mathrm{d}t = -\int_0^\infty x'\phi \mathrm{d}t, \quad \forall \phi \in C^1_c\left(\mathbb{R}_+, \mathbb{R}^n\right).$$

In this case, x' is called the derivative of x in the sense of distributions.

We recall some results that will be useful in our analysis (see Brezis 1983, for the proofs, pp. 119–148) about Sobolev space $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$:

- $W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space for the norm: $\|x\|_{W^{1,1}} = \|x\|_{L^1} + \left\|x'\right\|_{L^1}$.

- If $x \in W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$, there exists a unique continuous mapping $\widetilde{x}$ on $\mathbb{R}_+$ such that $x = \widetilde{x}$ almost everywhere.
- For all $\tau, \tau' \in \mathbb{R}_+$, $\widetilde{x}(\tau) - \widetilde{x}(\tau') = \int_{\tau'}^{\tau} x'(t)\mathrm{d}t$ and $\lim_{t\to\infty} \widetilde{x}(t) = 0$.

We consider a standard optimal control problem with an infinite horizon arising in dynamic models in continuous time:

$$\begin{array}{ll} \max & \int_0^\infty u(x(t), c(t))\,\mathrm{e}^{-rt}\mathrm{d}t \\ \text{subject to} & \\ & \dot{x}(t) = f(x(t), c(t)) \\ & x(0) = x_0 \end{array}$$

where $x(t) \in \mathbb{R}^n_+$ and $c(t) \in \mathbb{R}^m_+$.

We denote by E the space of functions from $\mathbb{R}_+$ to $\mathbb{R}^n$ such that $xe^{-rt} \in W^{1,1}(\mathbb{R}_+, \mathbb{R}^n)$. Let $\|x\|_E = \int_0^\infty \|x\|\,\mathrm{e}^{-rt}\mathrm{d}t + \int_0^\infty \|x'\|\,\mathrm{e}^{-rt}\mathrm{d}t$. By $L^1(\mathrm{e}^{-rt})$, we define the set of functions such that $x\mathrm{e}^{-rt} \in L^1$, for a given $r > 0$. Observe that $x \in E$ implies $\|x(t)\|\,\mathrm{e}^{-rt} \to 0$ when $t \to \infty$.

Next we make the following assumptions.

Assumption 1 $x \in E$ and $c \in L^1(\mathrm{e}^{-rt})$.

Assumption 2 f and u are continuous and the derivatives f_x, u_x are continuous.

Assumption 3 If x^*, c^* are optimal then $f_x(x^*, c^*) \in L^1(\mathrm{e}^{-rt})$ and $u_x(x^*, c^*) \in L^1(\mathrm{e}^{-rt})$.

The condition $x(0) = x_0$ must be understood in the sense that the unique continuous function $\widetilde{x}$ which is almost everywhere equal to x satisfies $\widetilde{x}(0) = x_0$.

Lemma 1 *Let L: $E \to \mathbb{R}_+$ be defined by $L(x) = x(0)$. The mapping L is Lipschitzian.*

Proof See Bonnisseau and Le Van (1996). □

Lemma 2 *Let D: $x(t) \to Dx(t) = \dot{x}(t)$. D is continuous from E into $L^1(\mathrm{e}^{-rt})$.*

Proof It is easy. □

Definition 2 *A trajectory $(x(t), c(t))$, $t \in [0, +\infty)$ is admissible if $x \in E$, $x \geq 0$, $c \in L^1_+(\mathrm{e}^{-rt})$, satisfy the constraints*

$$\dot{x}(t) = f(x(t), c(t))$$
$$x(0) = x_0$$

and if the integral in the objective function converges. A trajectory $(x^*(t), c^*(t))$ is an optimal solution if it is admissible and if the value of the objective function corresponding to any admissible trajectory is not greater than that of $(x^*(t), c^*(t))$.

The optimization problem under consideration can be recast in the following form $(\mathcal{P})$:

$$\max \quad U(x,c) = \int_0^\infty u\left(x(t), c(t)\right) \mathrm{e}^{-rt} \mathrm{d}t$$

subject to

$$\begin{aligned} Dx &= f(x,c) \\ Lx &= x_0, \end{aligned}$$

where U: $\left(E \cap L^1_+\left(\mathrm{e}^{-rt}\right)\right) \times L^1\left(\mathrm{e}^{-rt}\right) \to \mathbb{R} \cup \{-\infty\}$.

We now set an assumption, which is most crucial to our analysis:

Assumption 4 The optimal path is **supported** in the following sense. Let$(x^*(t), c^*(t))$ be an optimal solution. There exist multipliers $(a, q, \lambda) \in \mathbb{R}_+ \times L^\infty \times \mathbb{R}^n$ such that: $\forall x \in \left(E \cap L^1_+\left(\mathrm{e}^{-rt}\right)\right), \forall c \in L^1_+\left(\mathrm{e}^{-rt}\right)$,

$$\begin{aligned} &aU\left(x^*, c^*\right) - q\left(Dx^* - f\left(x^*, c^*\right)\right) - \lambda\left(Lx^* - x_0\right) \\ &\quad \geq aU(x,c) - q\left(Dx - f(x,c)\right) - \lambda\left(Lx - x_0\right). \end{aligned} \tag{1}$$

Notice that Assumption 4 is supposed to characterize the optimal paths: to each optimal solution, we assume that we can always assign multipliers, as usual associated with the objective function and constraints of the optimization problem, respectively, so that inequality (1) holds. Whether such multipliers do exist when an optimal path exists is addressed in section 4 for the Ramsey case: we don't argue here that such a property is inherent to optimality whatever the characteristics of the optimal control problem under study. Our first aim is to show that putting such an assumption in an appropriately defined Sobolev space very easily gives the Pontryagin conditions, including the transversality condition, at once. Proving the existence of the multipliers introduced in Assumption 4 is another task, which will be dealt with later.

Before showing this, some comments on Assumption 4 are necessary. As to the originality of our approach, it is fair to mention that an assumption like our Assumption 4 is not that far from the definition of *supported control trajectories* used in Peterson (1971) and more recently applied to a class of finite horizon optimal control problems by Carlson and Angell (1998). See for example Definition 5, p. 76, in Carlson and Angell. As in the pioneering work of Peterson, the latter authors easily prove for a class of undiscounted optimization problems with finite horizon that a control-trajectory which is feasible and supported is necessarily optimal (Theorem 6, p. 76). All these papers assume the existence of multipliers supporting the optimal trajectories in a sense fairly close to our inequality (1). However, beside the fact that the statement of inequality (1) depends on the optimization problem under study,[1] there are two main differences between this approach and ours. First of all, the literature mentioned just above follows different optimality criteria, namely overtaking optimality,[2] which has a lot to do with the undiscounted nature of the Ramsey problems under consideration. Moreover, we know by the

[1] It depends notably upon the boundary conditions of the problem under study.

[2] See for example, Carlson and Angell (1998 Definition 19, p. 85).

Halkin's counter-example that these undiscounted problems may not satisfy the usual transversality conditions. Therefore our framework and the associated optimality criterion (see Definition 2 above) are much better suited to the study of transversality conditions in economic problems.

Second, and more importantly, the treatment of the supporting function, q in our case, is far from similar, and it can, by no way, be the same because the involved functional spaces are completely different. Our application section provides an insightful constructive method to get the supporting function q, using the concept of order ideal in L^1 topology.

The next section gives the main results of the paper.

3 Main results

In this section, we shall show how our approach allows to derive properly and easily the Pontryagin conditions, and more importantly, it will be shown how it settles in a simple and natural way the problem of the necessity and sufficiency of the transversality condition for infinite horizon problems.

The next proposition can be viewed as a more accurate characterization of the supporting function q under assumptions 1 to 4.

Proposition 1 *Let Assumptions 1–4 be satisfied. Assume that $x^*(t) > 0$, $\forall t$. Then $\exists\, p \in L^1$ such that:*

$$au_x\left(x^*, c^*\right)\mathrm{e}^{-rt} + \dot{p}(t) + p(t) f_x\left(x^*, c^*\right) = 0, \tag{2}$$

in the sense of distributions.

Proof It is clear from (1) that one can write: $\forall x \in \left(E \cap L^1_+\left(\mathrm{e}^{-rt}\right)\right)$,

$$a\int_0^\infty \left[u\left(x^*, c^*\right) - u\left(x, c^*\right)\right]\mathrm{e}^{-rt}\mathrm{d}t - \int_0^\infty q(t)\left[Dx^* - Dx\right]\mathrm{e}^{-rt}\mathrm{d}t$$
$$+\int_0^\infty q(t)\left[f\left(x^*, c^*\right) - f\left(x, c^*\right)\right]\mathrm{e}^{-rt}\mathrm{d}t - \lambda\left[x^*(0) - x(0)\right] \geq 0. \tag{3}$$

Let $h(t) \in C_c^1\left(\mathbb{R}_+, \mathbb{R}^n\right)$. If $x^*(t) > 0$, $\forall t$, as $x^*(t)$ can be assumed to be continuous [recall that every element of the Sobolev space $W^{1,1}$ can be identified with a continuous function], we can choose μ sufficiently small such that $x(t) = x^*(t) + \mu h(t) \in E$. We obtain:

$$\int_0^\infty au_x\left(x^*, c^*\right)\mathrm{e}^{-rt}h(t)\mathrm{d}t - \int_0^\infty q(t)\mathrm{e}^{-rt}\dot{h}(t)\mathrm{d}t$$
$$+\int_0^\infty q(t)\mathrm{e}^{-rt} f_x\left(x^*, c^*\right)h(t)\mathrm{d}t = 0$$

and hence, with $p(t) = q(t)\mathrm{e}^{-rt} \in L^1$,

$$au_x\left(x^*, c^*\right)\mathrm{e}^{-rt} + \dot{p}(t) + p(t) f_x\left(x^*, c^*\right) = 0,$$

in the sense of distributions. □

It is easy to see that equation (2) is indeed the Pontryagin condition with respect to the state variable. Notice that the derivation of such a result is done in a very elementary way within our functional framework. The derivation of the necessary transversality condition is even more elementary:

Corollary 1 *Under the assumptions of Proposition 1, if an optimal solution* $(x^*(t), c^*(t))$ *exists, then necessarily* $p(t)\ e^{-rt} \in L^\infty$*, with* $p(t) = q(t)\ \mathrm{e}^{-rt}$*, defined in Proposition 1. In particular,* $\lim_{t\to\infty} p(t) = 0$*, and* $\lim_{t\to\infty} p(t)$ $(x^*(t) - x_0) = 0$.

The proof is trivial. In particular, the result of Long and Shimomura can be easily recovered. Indeed, knowing that $p(t)x^*(t) = q(t)\mathrm{e}^{-rt}x^*(t) \to 0$ and $p(t) \to 0$ as $t \to \infty$, we can derive directly $\lim_{t\to\infty} p(t)\,(x^*(t) - x_0) = 0$, as a necessary optimality condition. A further characterization of the multiplier $p(t)$ is also allowed.

Corollary 2 *If* $c^*(t)$ *is piecewise continuous then* $\dot{p}(t)$ *is piecewise continuous.*

Proof Since $p(t) \in L^1$, it follows from (2) that $\dot{p}(t) \in L^1$ and hence, $p(t)$ is continuous. This implies that $\dot{p}(t)$ is piecewise continuous. □

The next proposition gives the Pontryagin condition with respect to the control. Again, our approach allows for an almost immediate proof.

Proposition 2 *Let Assumptions 1–4 be satisfied. Assume that* $c^*(t)$ *is continuous and hence,* $\dot{x}^*(t)$ *is continuous. Then we have, for any* $c \in \mathbb{R}^m_+$*, any* $t \geq 0$*,*

$$\begin{aligned} &au\left(x^*(t), c^*(t)\right)\mathrm{e}^{-rt} + p(t) f\left(x^*(t), c^*(t)\right) \\ &\quad \geq au\left(x^*(t), c\right)\mathrm{e}^{-rt} + p(t) f\left(x^*(t), c\right). \end{aligned} \tag{4}$$

Proof From (1), it can be noted that, $\forall z \in L^1_+\left(\mathrm{e}^{-rt}\right)$,

$$\begin{aligned} &a\int_0^\infty \left[u\left(x^*, c^*\right) - u\left(x^*, z\right)\right]\mathrm{e}^{-rt}\mathrm{d}t \\ &\quad + \int_0^\infty p(t)\left[f\left(x^*, c^*\right) - f\left(x^*, z\right)\right]\mathrm{d}t \geq 0. \end{aligned} \tag{5}$$

Assume on the contrary, by continuity,

$$\begin{aligned} &au\left(x^*(t), c^*(t)\right)\mathrm{e}^{-rt} + p(t) f\left(x^*(t), c^*(t)\right) \\ &\quad < au\left(x^*(t), c\right)\mathrm{e}^{-rt} + p(t) f\left(x^*(t), c\right) \end{aligned}$$

in some interval I around t with some positive constant $c \geq 0$. Let $c'(t) = c^*(t)$, $t \notin I$ and $c'(t) = c$ when $t \in I$. Note that $c'(t) \in L^1_+\left(\mathrm{e}^{-rt}\right)$. However, (5) is not satisfied leading to a contradiction. □

We now move to the sufficiency part and prove among others that the previously derived transversality condition is sufficient for optimality under some conditions which are known in the optimization literature (see for example, Carlson et al. 1991). To this end, we use the Hamiltonian concept.

Assumption 5 Define the Hamiltonian

$$H(x, c, p, t) = u\left(x(t), c(t)\right) \mathrm{e}^{-rt} + p(t) f\left(x(t), c(t)\right).$$

Suppose that, $\max_{c \geq 0} H(x, c, p, t)$ is concave in x and

$$H\left(x^*, c^*, p, t\right) \geq H\left(x^*, c, p, t\right), \quad \forall c \geq 0.$$

The next proposition shows the sufficiency of the transversality condition when Assumption 5 is added to our assumptions set.

Proposition 3 *Under Assumptions 1–3 and Assumption 5, a sufficient condition for* $(x^*(t), c^*(t))$ *to be optimal is*

$$p(t) e^{rt} \in L^{\infty}.$$

Proof By Assumption 5, the following holds for every $T > 0$:

$$\int_0^T u\left(x^*(t), c^*(t)\right) \mathrm{e}^{-rt} \mathrm{d}t - \int_0^T u\left(x(t), c(t)\right) \mathrm{e}^{-rt} \mathrm{d}t \geq p(T)\left(x(T) - x^*(T)\right).$$

By assumption, $p(t)e^{rt} \in L^{\infty}$. We then have:

$$\begin{aligned} \left|p(T)\left(x(T) - x^*(T)\right)\right| &\leq \|p(T)\| e^{rT} \left[\|x(T)\| \mathrm{e}^{-rt} + \left\|x^*(T)\right\| \mathrm{e}^{-rt}\right] \\ &\leq K\left[\|x(T)\| \mathrm{e}^{-rt} + \left\|x^*(T)\right\| \mathrm{e}^{-rt}\right]. \end{aligned}$$

Since $x \in E$ and $x^* \in E$, we get $K\left[\|x(T)\| \mathrm{e}^{-rt} + \|x^*(T)\| \mathrm{e}^{-rt}\right] \to 0$ as $T \to \infty$. That ends the proof. □

Along this section, we have shown how the Sobolev space topology choice simplifies greatly the analysis of the necessary and/or sufficient transversality condition. While the sufficiency part of our analysis relies on standard conditions, the necessary conditions part is not that standard, at least in the literature of transversality conditions. Clearly, the crucial part of our analysis is Assumption 4. We show in the next section how this assumption is checked in the Ramsey-like models using common tools in general equilibrium theory.

4 Application to the Ramsey model

We consider the following usual type of Ramsey model:

$$\begin{array}{ll} \max & \int_0^\infty u\left(c(t)\right)\mathrm{e}^{-rt}\mathrm{d}t \\ \text{subject to} & \\ & c(t)+\dot{x}(t)\leq f\left(x(t)\right)-\delta x(t) \\ & c(t)\geq 0, \forall t \\ & x(t)\geq 0, \forall t \\ & x(0)=x_0>0, \text{ is given.} \end{array}$$

under the following assumptions.

Assumption 6 u is C^1, strictly concave, increasing with $u^{'}(0)=+\infty$.

Assumption 7 f is C^1, strictly concave, increasing with $f^{'}(0)>\delta$, $f^{'}(0)<\infty$, $f^{'}(\infty)=0$.

Proposition 4 *The optimal solution* $(x^*(t),c^*(t))$ *satisfy* $x^*\in W^{1,1}\cap L^1_+$ *and* $c^*\in L^1_+\left(\mathrm{e}^{-rt}\right)$.

Proof See Askenazy and Le Van (1999). □

In accordance with this proposition, we can use $W^{1,1}\cap L^1_+$ as the state space and L^1_+ as the control space. Let $X=\left(W^{1,1}\cap L^1_+\right)\times L^1_+$. The problem becomes:

$$\begin{array}{ll} \max & U(x,c)=\int_0^\infty u\left(c(t)\right)\mathrm{e}^{-rt}\mathrm{d}t \\ \text{subject to} & \\ & g(x,c)\geq 0 \\ & Lx=x_0, \end{array}$$

where $g(x,c)=f(x)-\delta x-c-Dx$ and $Lx=x(0)$. Note that g takes values in L^1 and L^1_+ has an empty interior. Hence, the direct application of the theorem V.3.1 of Hurwicz (1958) is not possible for proving the existence of the multipliers $(a,q,\lambda)\in\mathbb{R}_+\times L^\infty\times\mathbb{R}^n$ associated with this problem. We then use the same approach as Mas-Colell and Zame (1991) and Dana et al. (1997). We consider an order ideal which is dense in the original space. There we have the positive orthant of the order ideal with a nonempty interior for its lattice norm.

It is well known (see for example, Askenazy and Le Van, 1999, Proposition 5) that there exist $\alpha>0$, $\alpha'>0$ such that the optimal consumption path satisfies: $\alpha'\geq c^*(t)\geq\alpha$, $\forall t\geq 0$, and there exist $\beta>0$, $\beta'>0$ such that the optimal capital path satisfies $\beta'\geq x^*(t)\geq\beta$, $\forall t\geq 0$, Let $\bar{c}=c^*$ and $I\left(\bar{c}\right)=\left\{y\in L^1\text{: }\exists\mu>0\text{ s.t. }|y|\leq\mu\bar{c}\right\}$.

The ideal $I\left(\bar{c}\right)$ is dense for both the L^1 topology and for the weak topology (see Aliprantis et al. 1990, pp. 103,104). We define on $I\left(\bar{c}\right)$ the norm $\|.\|_{\bar{c}}$:

$$\|y\|_{\bar{c}}=\inf\left\{\mu>0:|y|\leq\mu\bar{c}\right\}.$$

One can verify that the positive orthant of $I\left(\bar{c}\right)$, $I_+\left(\bar{c}\right)$ has nonempty interior for the topology defined by $\|.\|_{\bar{c}}$. More precisely, $\bar{c} \in intI\left(\bar{c}\right)$. Similarly, one can define $I(\bar{x})$ with $\bar{x} = x^*$ and $x^* \in intI(\bar{x})$. It is obvious that $I(\bar{c}) = I(\bar{x})$.

As u is increasing by Assumption 6, along an optimal path, we have that $g(x^*, c^*) = 0$, i.e. $g(x^*, c^*) \in I_+\left(\bar{c}\right)$. Consider the problem:

$$\begin{array}{ll} \max & U(x,c) \\ \text{subject to} & \\ & g(x,c) \in I_+\left(\bar{c}\right) \\ & Lx = x_0, \end{array}$$

where one can now apply the Theorem V.3.1 of Hurwicz (1958) and obtain:

$$\begin{aligned} &\exists (a,q,\lambda) \in \mathbb{R}_+ \times I_+\left(\bar{c}\right)' \times \mathbb{R} \text{ s.t.} \\ &aU\left(x^*,c^*\right) + qg\left(x^*,c^*\right) + \lambda\left(Lx^* - x_0\right) \\ &\geq aU(x,c) + qg(x,c) + \lambda\left(Lx - x_0\right), \end{aligned} \tag{6}$$

$\forall x, \forall c,$ such that $g(x,c) \in I(\bar{c})$.

Now we shall prove that q is a continuous linear form on $I\left(\bar{c}\right)$ for the L^1-*norm* topology. Since $I\left(\bar{c}\right)$ is dense in L^1, it extends to a continuous form on L^1 for the L^1-*norm* topology. To this end, we follow Dana et al. (1997) and utilize the notion of properness.

Since $c_t^* \geq \alpha > 0, \forall t$, it is clear that $u'\left(c^*(t)\right) \in L^\infty$. From Le Van (1996), U is proper at c^*. Hence, there exists an open **solid** neighborhood of 0, denoted by A and a vector $v \in L^1_+$ such that $\forall \mu > 0$, small enough,

$$U\left(x, c^* + \mu(v+z)\right) > U\left(x,c^*\right) \quad \text{if } z \in A \quad \text{and} \quad \text{if } c^* + \mu(v+z) \in L^1_+.$$

Actually, we can take $v(t) = 1, \forall t$, and

$$A = \left\{ x \in L^1 : \int_0^{+\infty} u'(c_t^*)|x(t)|\mathrm{e}^{-rt}\mathrm{d}t < \int_0^{+\infty} u'(c_t^*)\mathrm{e}^{-rt}\mathrm{d}t \right\}.$$

It is obvious that A is an open solid set of L^1 and contains 0. Let $c^* + \mu(1+z) \geq 0$ with $\mu > 0$ and $z \in A$. We have

$$\lim_{\mu\to 0} \frac{U(x,c^*{+}\mu(1+z)) - U(x,c^*)}{\mu} = \lim_{\mu\to 0} \int_0^{+\infty} \frac{u(c^* + \mu(1+z)) - u(c^*)}{\mu}\mathrm{e}^{-rt}\mathrm{d}t$$

$$= \int_0^{+\infty} u'(c^*)(1+z)\mathrm{e}^{-rt}\mathrm{d}t \geq \int_0^{+\infty} u'(c^*)\mathrm{e}^{-rt}\mathrm{d}t - \int_0^{+\infty} u'(c^*)|z|\mathrm{e}^{-rt}\mathrm{d}t > 0.$$

Thus $U(x, c^* + \mu(1+z)) > U(x,c^*)$ for any $\mu > 0$ small enough.

Now, let $y \in A \cap I_+\left(\bar{c}\right)$. There exists $\mu > 1$ such that $0 \leq y \leq \mu\,\bar{c} = \mu c^*$. Define $z = (1/\mu)y$. We have $c^* + (1/\mu)(v - y) \geq 0$. By applying the inequality (6):

$$\begin{aligned}
&aU\left(x^*, c^*\right) + qg\left(x^*, c^*\right) + \lambda\left(Lx^* - x_0\right)\\
&\geq aU\left(x^*, c^* + \frac{1}{\mu}(v - y)\right) + qg\left(x^*, c^* + \frac{1}{\mu}(v - y)\right) + \lambda\left(Lx^* - x_0\right)\\
&= aU\left(x^*, c^* + \frac{1}{\mu}(v - y)\right) + q\left(g\left(x^*, c^*\right) - \frac{1}{\mu}(v - y)\right) + \lambda\left(Lx^* - x_0\right)
\end{aligned}$$

together with the properness condition lead us to obtain that

$$qv \geq qy.$$

On the other hand, since $c^* + (1/\mu)(v + y) \geq 0$, we have also

$$qv \geq -qy.$$

Now, let $y \in A \cap I\left(\bar{c}\right)$. y^+ and y^- belong to $A \cap I_+\left(\bar{c}\right)$. We have $qy^+ \leq qv$ and $-qy^- \leq qv$ so that $qy \leq 2qv$. We have proved that the linear form q is bounded from above in an open neighborhood of 0. Therefore, q is continuous on $I\left(\bar{c}\right)$ with the initial topology. Since $I\left(\bar{c}\right)$ is dense in L^1, q has a unique extension in $\left(L^1\right)' = L^\infty$.

We now show that inequality (6) also holds for $g(x, c) \in L^1$.

First, since c^* is in the interior of $I_+(\bar{c})$, we have one of the first order conditions:

$$q(t) = au'(c^*(t)) \tag{7}$$

Since x^* is also in the interior of $I_+(\bar{x}) = I_+(\bar{c})$, we have another first order condition

$$\lambda h(0) + \int_0^\infty q(t)(f'(x^*(t))h(t) - \delta h(t) - Dh(t))\mathrm{e}^{-rt}\mathrm{d}t = 0 \tag{8}$$

for any C^1 - function h with compact support in $\mathbb{R}$ such that there exists $\zeta > 0$ which satisfies $x^* + \zeta h \in I(\bar{c})$. Since h and Dh are bounded, we have hence $g(x^* + \zeta h, c^*) \in I(\bar{c})$. From Brezis (1983), Theorem VIII.6, if h is in $W^{1,1}$ then there exists a sequence of functions $\{h_n\}$, C^1, with compact support in $\mathbb{R}$ which converge to h in $W^{1,1}$. For any n, there exists $\zeta_n > 0$ such that $x^* + \zeta_n h_n \in I(\bar{c})$. Thus h_n satisfies (8). From Lemma 1 and Lemma 2, Relation (8) holds therefore for any $h \in W^{1,1}$. Now, the two relations (7) and (8) imply that inequality (6) holds whether $g(x, c)$ is in $I(\bar{c})$ or not.

References

Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Existence and optimality of competitive equilibria. Berlin Heidelberg New York: Springer 1990

Aliprantis, C.D., Cornet, B., Tourky, R.: Economic equilibrium: Optimality and price decentralization. Positivity **6**, 205–241 (2002)

Araujo, A., Monteiro, P.K.: General equilibrium with finitely many goods: the case of separable utilities. Equilibrium and Dynamics: Essays in honor of David Gale. London: McMillan 1989

Askenazy, P., Le Van, C.: A model of optimal growth strategy. J Econ Theory **85**(1), 24–51 (1999)

Benveniste, L.M., Scheinkman, J.A.: Duality theory for dynamic optimization models of economics: The continuous time case. J Econ Theory **27**, 1–19 (1982)

Bonnisseau, J-M., Le Van, C.: On the subdifferential of the value function in economic optimization problems. J Math Econ **25**, 55–73 (1996)

Brezis, H.: Analyse Fonctionelle: Théorie et applications. In: Ciarlet, P.G., Lions, J.L. (eds.) Collection Mathematiques appliquees pour la maitrise. Paris: Masson 1983

Carlson, D., Angell, T.: Turnpike property for neutral systems. Nonlinear Stud **5**, 69–94 (1998)

Carlson, D., Haurie, A., Leizarowitz, A.: Infinite horizon optimal control: deterministic and stochastic systems, 2nd ed. Berlin Heidelberg Newyork: Springer 1991

Chichilnisky, G.: Nonlinear functional analysis and optimal economic growth. J Math Anal Appl **61**, 504–520 (1977)

Dana, R-A., Le Van, C., Magnien, F.: General equilibrium in asset markets with or without short-selling. J Math Anal Appl **206**, 567–588 (1997)

Kamien, M., Schwartz, N.: Dynamic optimization. Amsterdam: North-Holland 1991

Kamihigashi, T.: Necessity of transversality conditions for infinite horizon problems. Econometrica **69** (4), 995–1012 (2001)

Hadley, G., Kemp, M.: Variational methods in economics. Amsterdam: North-Holland 1973

Halkin, H.: Necessary conditions for optimal problems with infinite horizon. Econometrica **42** (2), 267–272 (1974)

Hurwicz, L.: Programming in linear spaces. In: Arrow, K.J., Hurwicz, L. Uzawa, H. (eds.) Studies in Linear and non-linear programming. Stanford Mathematical Studies in the Social Sciences, II. Stanford: Stanford university press 1958

Le Van, C.: Complete characterization of Yannelis-Zame and Chichilnisky-Kalman-Mas-Colell propernes conditions on preferences for separable concave functions defined in L_+^p and L^{p^*}. Econ Theory **8**, 155–166 (1996)

Long, N. V., Shimomura, K.: A note on transversality conditions. RIEB Discussion Paper Series **144**, Kobe University (2003)

Mas-Colell, A., Zame, W.: Equilibrium theory in infinite dimensional spaces. In: Hildebrandt, W., Sonnenschein, H., (eds.) Handbook of Mathematical Economics vol. IV. Amsterdam: North-Holland 1991

Michel, P.: On the transversality condition in infinite horizon problems. Econometrica **50**(4), 975–985 (1982)

Peterson, D.W.: A sufficient maximum principle. IEEE Trans Autom Control **16**(1), 85–86 (1971)

Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E. F.: The mathematical theory of optimal processes. New York: Interscience publishers, Wiley 1962

Ye, J. J.: Nonsmooth maximum principle for infinite horizon problems. J Optim Theory Appl **76**, 485–500 (1993)